Functions and Calculus

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What is a function?

A mathematical function takes an input argument and provides a unique output argument

$$y = f(x)$$

$$e.g.$$

$$f(x) = 2 * x, \;\; f(x) = 3 * x^2 + 4, \text{etc.}$$

x and f(x) can mean anything, e.g. space, time, money, age etc.

A function may not be defined for all input arguments *x* (e.g. 1/*x*)

There may not be an *x* to every output argument *y* (e.g. *sine(x)* is always <=1)

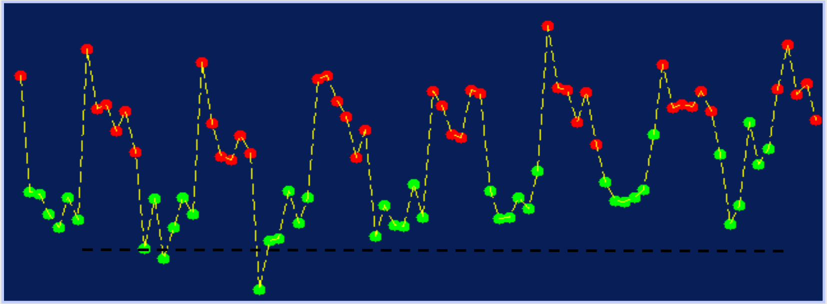
There may be multiple *x* for a given output argument *y* (e.g. $2^2=(-2)^2=4$)

But functions to not produce multiple outputs for the same input x

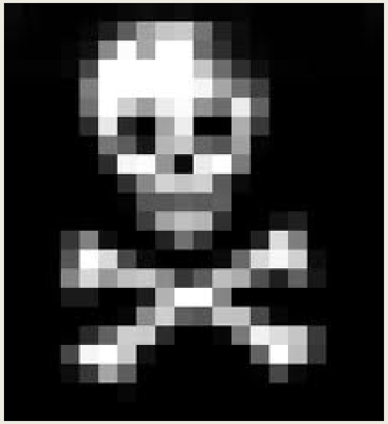
Discretisation: Turning functions into vectors and matrices

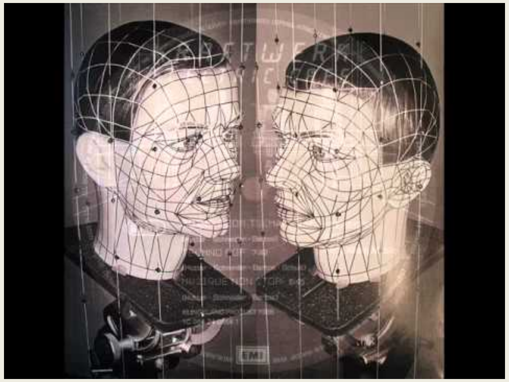
EEG/MEG

voxel timeseries



etc.





The density of sampling determines the accuracy and speed of any further operations

Popular functions: Polynomials

Important for regression, power laws, statistics, model fitting etc.

Most common with positive exponents

$$f(x) = \sum_i a_i x^i$$

$$e.g.$$

$$f(x) = 2 * x$$

$$f(x) = 2 + x + 3 * x^2 + 10 * x^7$$

But why not try negative exponents:

$$x^{-i} = \frac{1}{x^i}$$

Or rational exponents:

$$x^{\frac{p}{q}} = \sqrt[q]{x^p}$$

Examples

Popular functions: sine, cosine and cousins

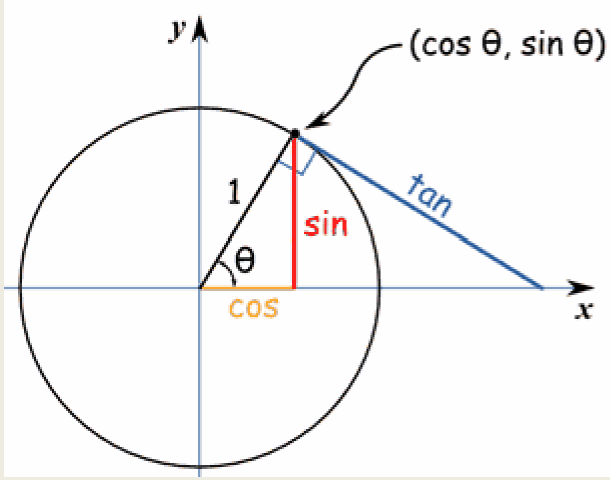
Important for oscillations, filters, Fourier Transform etc.

$$f(x) = a * \sin(b * x + c)$$

$$f(x) = a * \cos(b * x + c)$$

$$a : \text{amplitude}, b : \text{frequency}, \text{c} : \text{phase}$$

$$\text{Then there is the tangens}: \ f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$



θ takes the role of x, the lengths of the coloured lines are the function outputs

Inverse of sine and cosine: arcsine and arccosine

Examples

Popular functions: logarithm and exponential functions

Important for growth and decay processes, likelihoods, Bayesian estimation etc.

$$\text{Exponential}: f(x) = \exp(x) = e^x, \; e \approx 2.72$$

The inverse of the exponential is the "natural" logarithm:

$$\log\left(e^x\right) = x$$

This can be done with any "base", e.g.: $f(x) = 10^x$

Then logarithm to base 10:

$$\log_{10}\left(10^x\right) = x$$

In Matlab, "log" is the natural logarithm (sometimes "ln" in the literature), i.e.

$$\log(\mathrm{x}) = \ln(\mathrm{x}) = \log_{\mathrm{e}}(x)$$

But the exponential function is special (more later)

Examples

Combining functions

Gaussian bell curve:

$$f(x) = \exp(a*(x-x_0)^2) = e^{a*(x-x_0)^2}$$

"Speeding up" and "slowing down" oscillations:

$$f(x) = \cos(\exp(a*x)), \quad f(x) = \cos(\exp(-a*x))$$

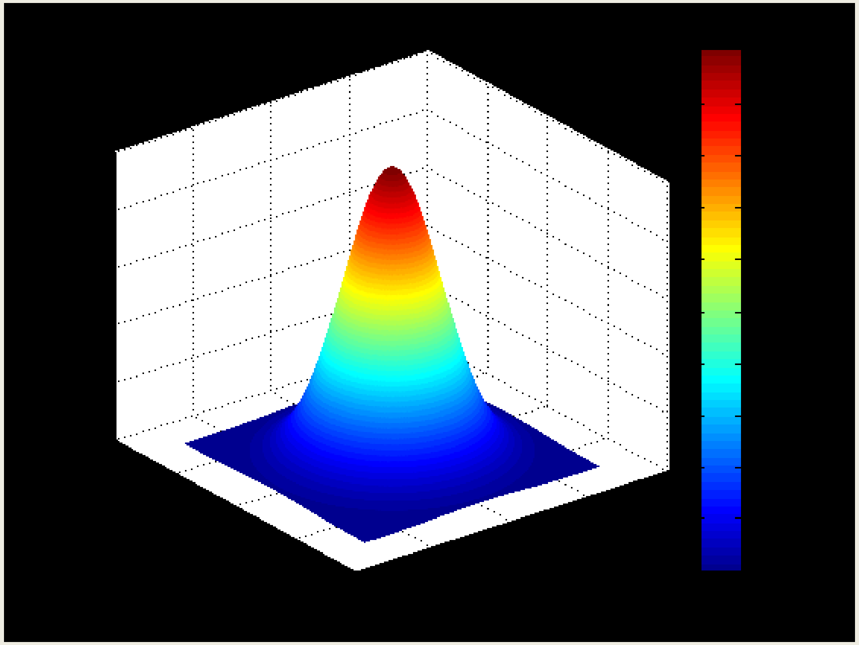
Multiplying functions in Matlab results in element-wise multiplication of vectors

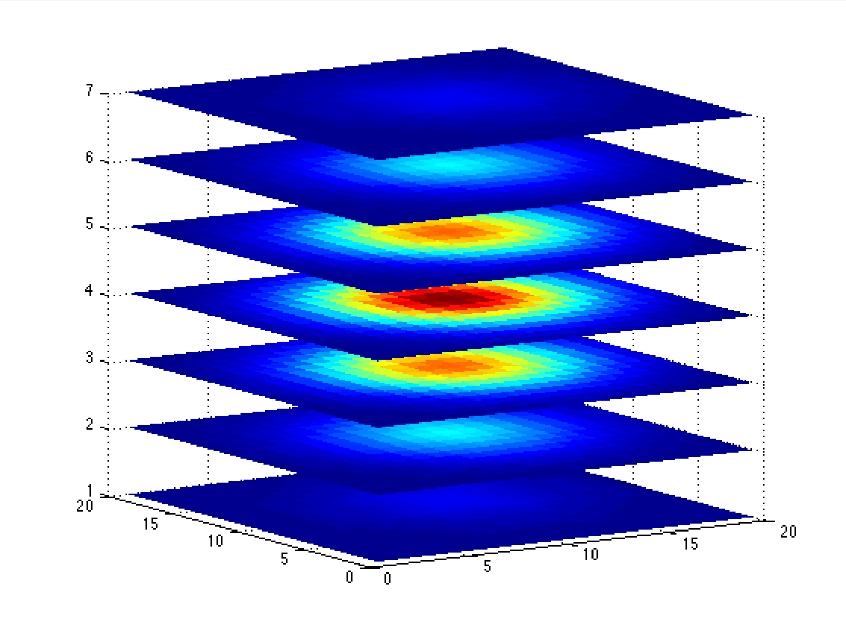
$\exp(x)*\sin(x)$ -> **exp_x.*sin_x** (dampened oscillation)

Examples

Functions in multiple dimensions

(only quickly)





Examples

Complex Numbers

$$i = \sqrt{-1}$$

$$therefore:$$

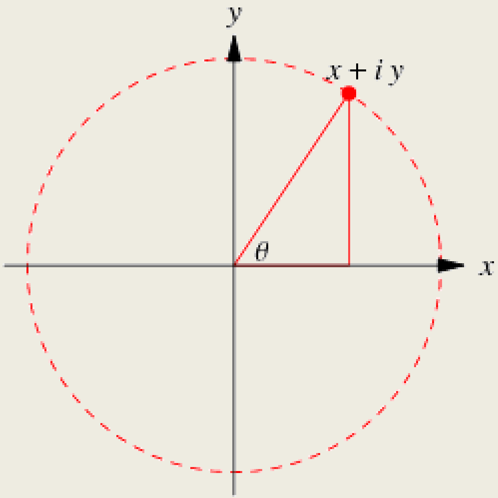
$$i * i = -1$$

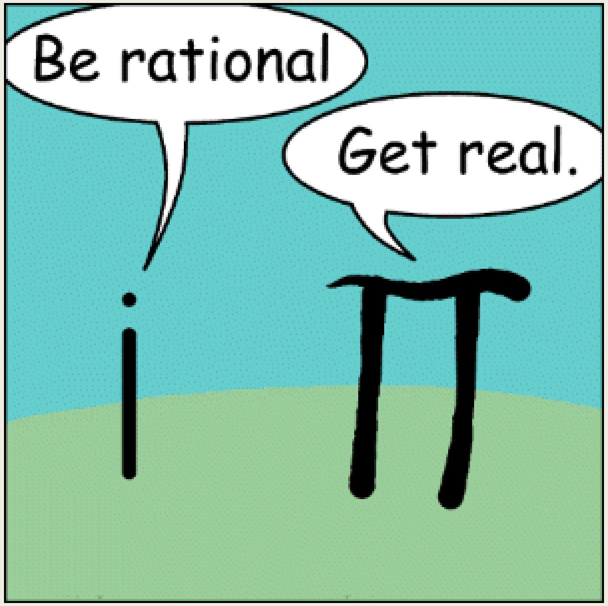
Numbers can be represented with a "real" and "imaginary" part:

$$x + y*i$$

Functions can operate on both real and imaginary parts simultaneously

e.g. $(1 + 2*i)^2 = 4*i - 3$





Complex Numbers

Examples

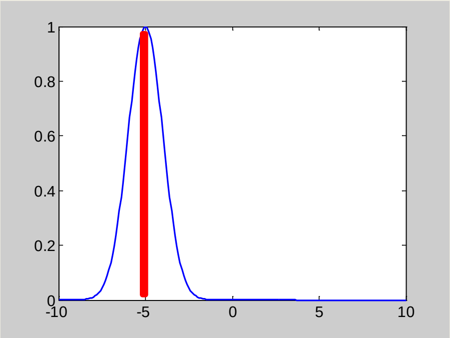
Define your own function

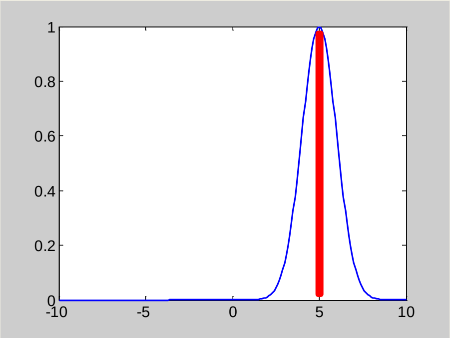
Examples

Convolution

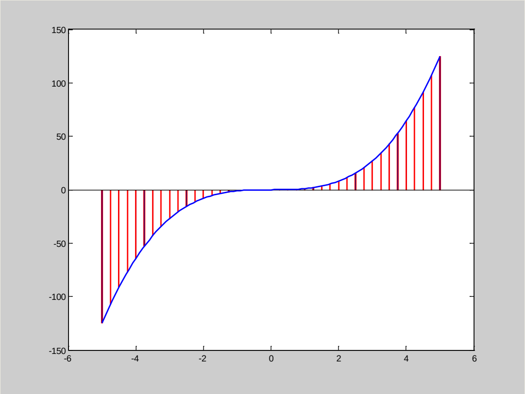
Describes many linear systems, physical laws, equations etc.

Imagine a filter which – whenever the input is just an infinitely short peak – outputs a blurred Gaussian bell curve (“point-spread”).



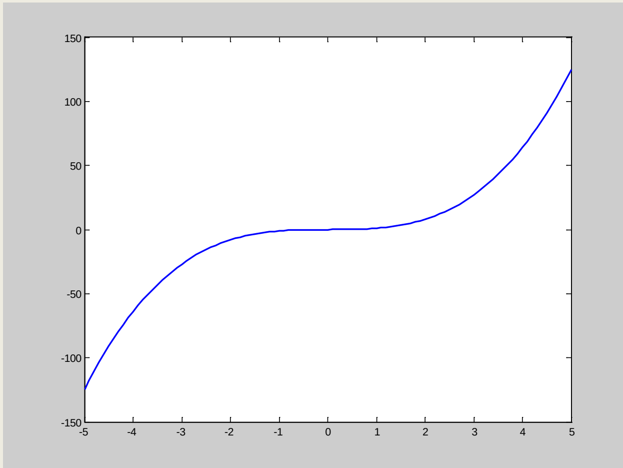


More complex functions can be described as a weighted sum of an infinite number of infinitely small peaks (“delta functions”):

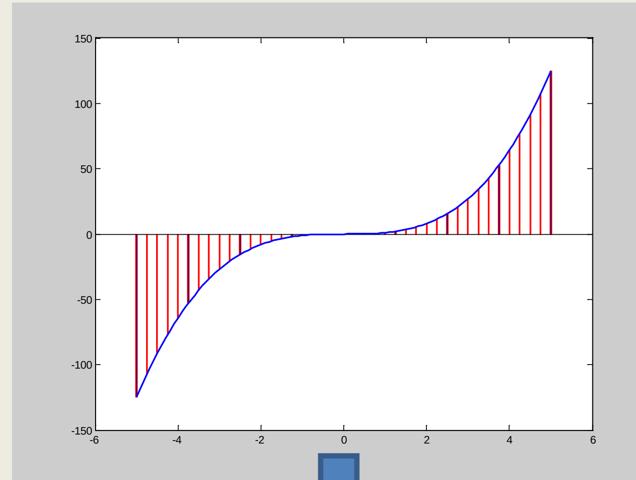


Convolution

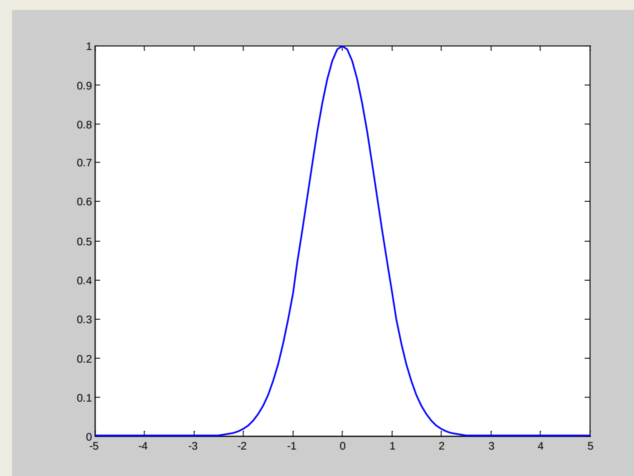
The original function…



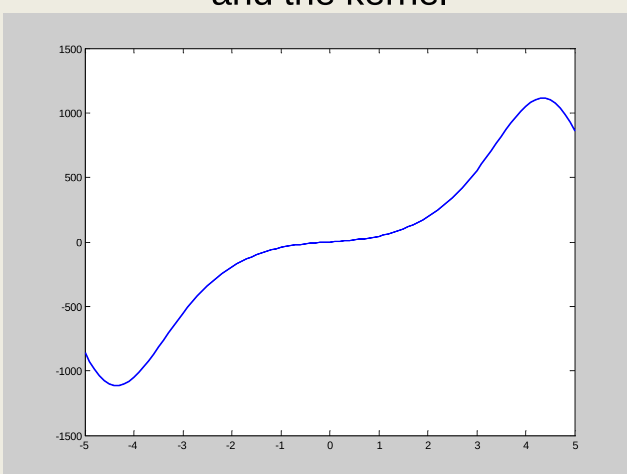
…described as an infinite sum of delta peaks



Each peak is blurred by the “convolution kernel”

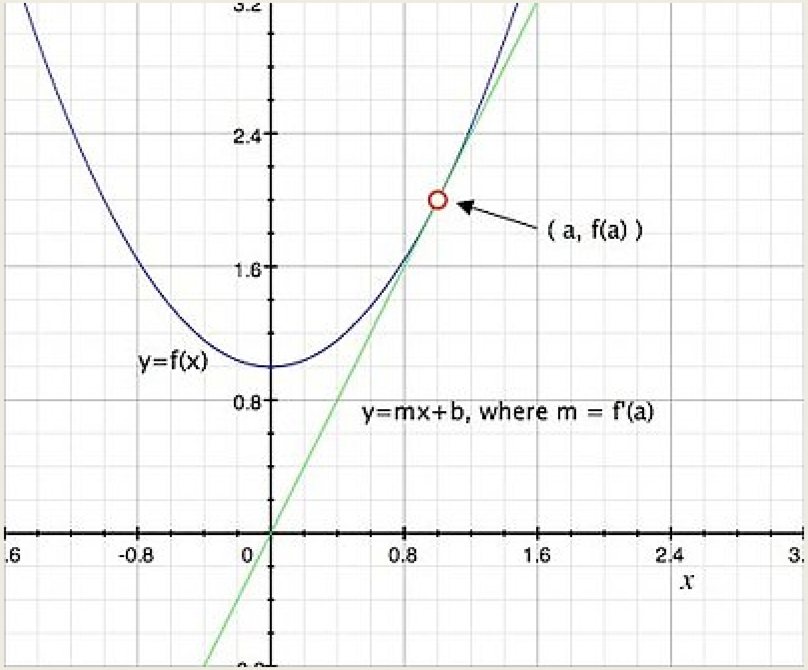


The result is the sum of the blurred peaks - i.e. the convolution of the original function and the kernel



Examples

Differentiation



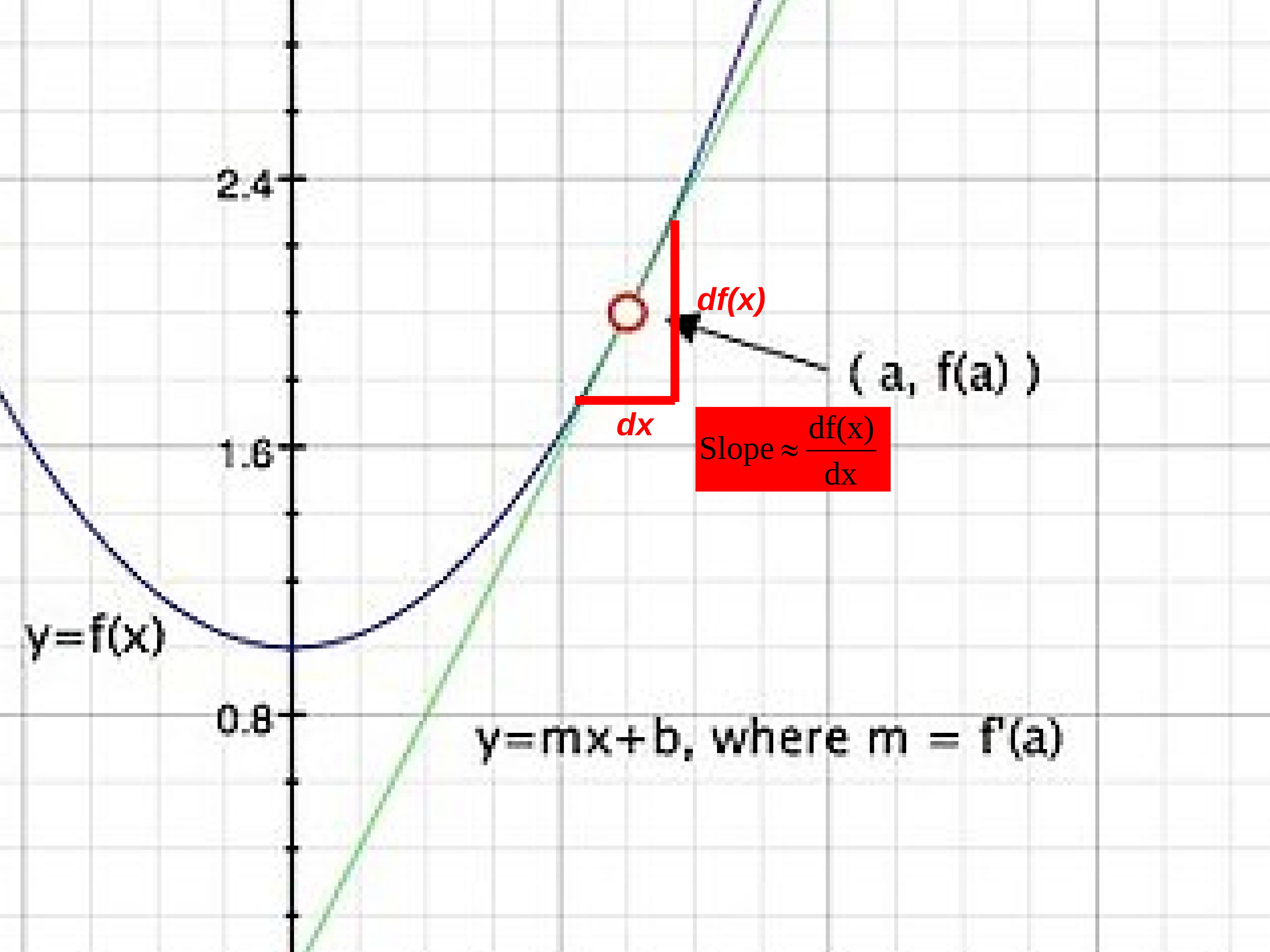
The derivative describes the local rate of change of a function, or the slope of a line that best approximates the function in one point

Notations:

$$f'(x)$$

$$\frac{d}{dt} f(t) \text{ (e.g. for time)}, \quad \frac{d}{dx} f(x) \text{ (e.g. for space)}, \quad \dot{f}(t) \text{ (common for time)}$$

For example: if t is time, $f(t)$ is your distance from some point on a line, then $f'(t)$ is your speed at time t


2.4
1.6
0.8
df(x)
(a, f(a))
dx
Slope ≈ df(x)/dx
y=f(x)
y=mx+b, where m = f'(a)

Differentiation

There is a command **diff()** in Matlab (assumes steps of 1 in x-direction), but it's of course even better to know the derivatives exactly, for example:

f(x) = exp(x)	f'(x) = exp(x)	f'(exp(a*x)) = a*exp(a*x)
f(x) = sin(x)	f'(x) = cos(x)	f'(sin(a*x)) = a*cos(a*x)
f(x) = cos(x)	f'(x) = -sin(x)	f'(cos(a*x)) = -a*sin(a*x)

f(x) = 1	f'(x) = 0
f(x) = x	f'(x) = 1
f(x) = x^2	f'(x) = 2*x
f(x) = x^a	f'(x) = a*x^{a-1}

Differential Equations

If we don't know a function, but we know it's relationship to its derivate(s), then this can be expressed as a differential equation:

$$\text{e.g. } f'(x) = f(x) + x^2$$

Lots of physical/engineering problems can be formulated like this. The problem then is to find the solution that solves this equation.

Differential Equations

Example:

You have a number of rabbits *x*.
They are breeding like you would expect.
The more rabbits, the more breeding.
In fact, the rate of increase of *x* is the number of new rabbit babies per time (*dx/dt*), which is proportional to the number of rabbits *x*.
Ergo:

$$f'(x) = a * f(x)$$

$$\text{with solution}: f(x) = \exp(a * x) \quad (\text{see previous slide})$$

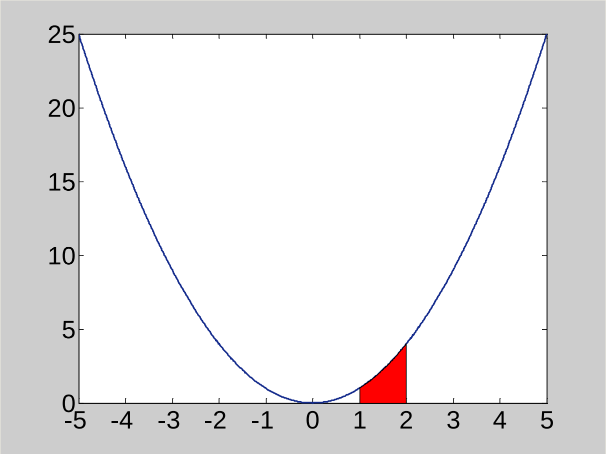
a then reflects something like the “productivity” of the rabbit population.

Integration

$$F(x) = \int f(x)dx$$

The integral $\int_a^b f(x)dx = \mathrm{F(b) - F(a)}$ provides the area under the curve f(x) between a to b

e.g. $\int_1^2 x^2 dx$

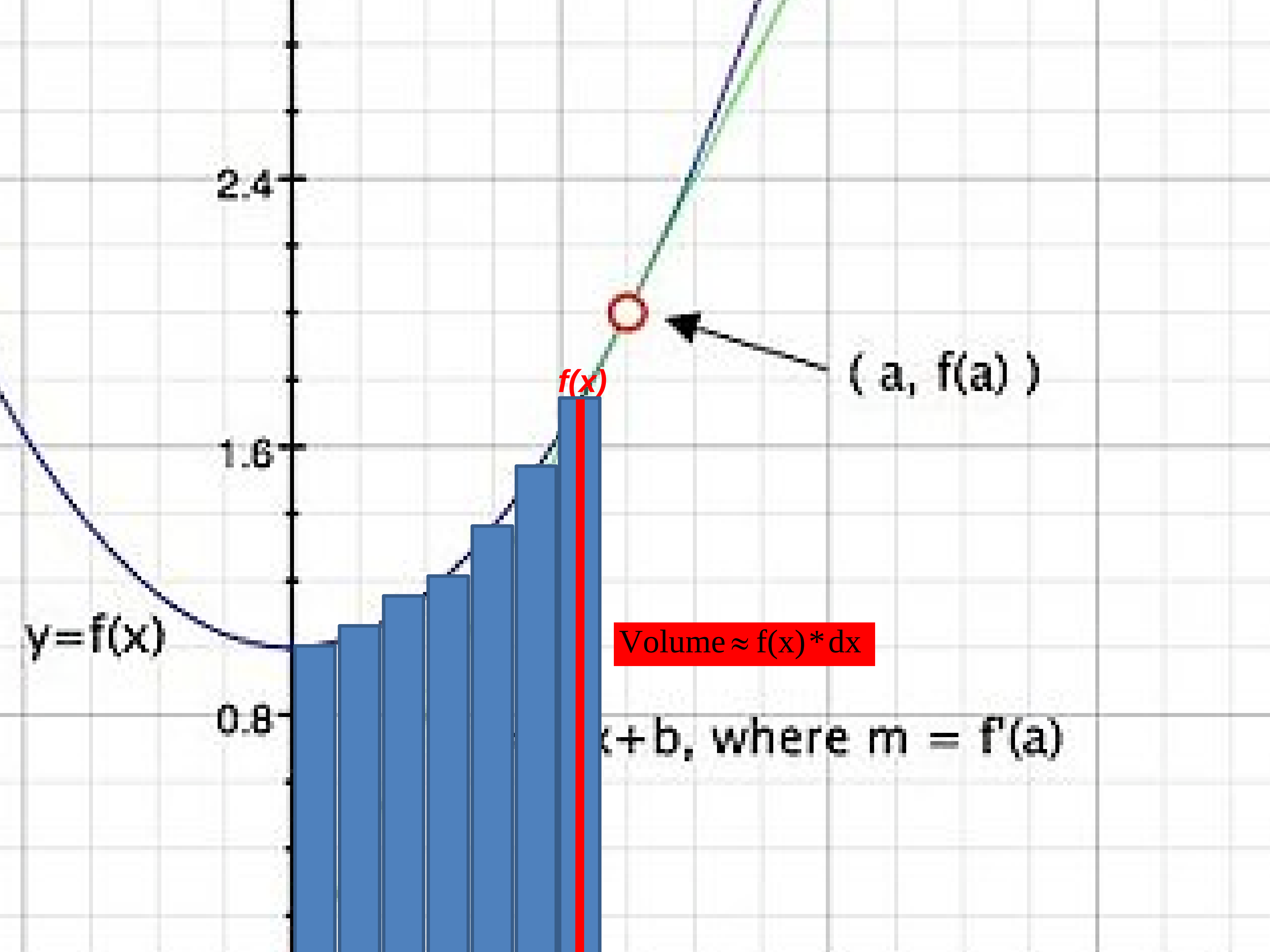


$$= \frac{7}{3} = 2\frac{1}{3} = 2.\overline{3}$$

Integration and differentiation are the reverse of each other:

$$\mathrm{g(x)} = \int f(x)dx \quad \text{then} \quad \mathrm{g'(x) = f(x)}$$

$$\mathrm{h(x) = f'(x)} \quad \text{then} \quad \int \mathrm{h(x)dx = f(x) + c}, \ \text{with c an arbitrary constant}$$


2.4
1.6
0.8
(a, f(a))
f(x)
y=f(x)
Volume ≈ f(x)*dx
+b, where m = f'(a)

Integration

For most common functions, integration can be done analytically:

If we know

$$\int x^2 dx = \frac{1}{3}\mathrm{x}^3$$

then

$$\int_{\mathrm{a}}^{\mathrm{b}} x^2 dx = \frac{1}{3}\mathrm{b}^3 - \frac{1}{3}\mathrm{a}^3$$

e.g. $\displaystyle\int_1^2 x^2 dx = \frac{1}{3}2^3 - \frac{1}{3}1^3 = \frac{7}{3}$

Integration

f(x) = exp(x)	f'(x) = exp(x)	$\int \exp(x)dx = \exp(x)$
f(x) = sin(x)	f'(x) = cos(x)	$\int \cos(x)dx = \sin(x)$
f(x) = cos(x)	f'(x) = -sin(x)	$\int \sin(x)dx = -\cos(x)$

f(x) = 1	f'(x) = 0	$\int 0dx = 0$
f(x) = x	f'(x) = 1	$\int 1dx = x$
f(x) = x^2	f'(x) = 2*x	$\int 2 * xdx = x^2$
f(x) = x^a	f'(x) = $a*x^{a-1}$	$\int x^a = \frac{1}{a}x^{a+1}$

Examples